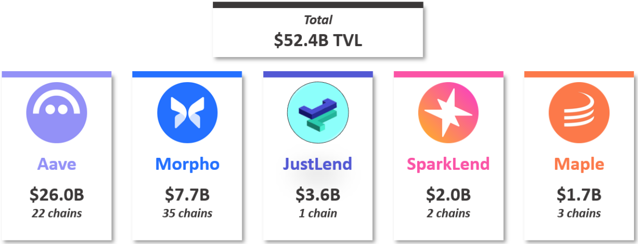
Mini-course on Quantitative Decentralized Finance

Bachelier Seminar · Institut Henri Poincaré, Paris

Lending-Borrowing Protocols

Risk Modeling in Aave-v3

DeFi Lending Overview



Total
$52.4B TVL

Aave	Morpho	JustLend	SparkLend	Maple
$26.0B	**$7.7B**	**$3.6B**	**$2.0B**	**$1.7B**
22 chains	*35 chains*	*1 chain*	*2 chains*	*3 chains*

Aave protocol:

- Originally ETHLend in 2017;
- Version 3 launched in January 2023;
- Version 4 launched this month.

Outline

Outline

Aave Protocol Architecture
- Protocol Structure
- Interest Rates Strategy
- Over-Collateralization & Liquidation

Mathematical Modelling of Liquidation
- Probability of Liquidation
- Impact of Liquidations on Returns
- Fast Sampling Scheme for Returns
- Application to Portfolio Optimization

Protocol Structure

Liquidity Reserves (one per token)

Lenders

Deposit assets
Earn interest

Borrowers

Post collateral
Borrow assets
Earn & Pay interests

Liquidators

Close risky
Positions
Earn bonus

Protocol Structure

65

Available Tokens

29,7K

Distinct Active Addresses

$22,7B

Deposited

$13,4B

Borrowed

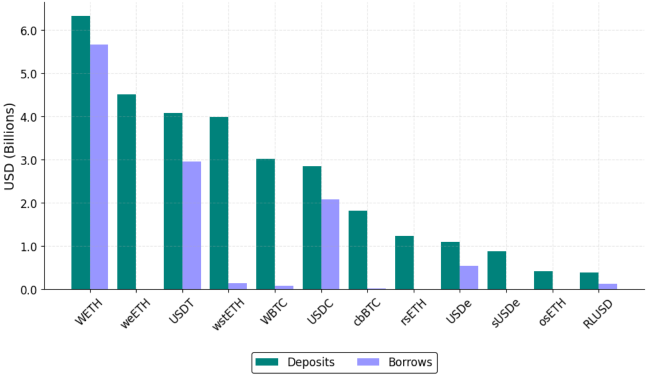


Figure: Deposits and Borrows per Reserve. Snapshot date: 2026-04-12.

Interest Rates Strategy

- **Utilization ratio** is a critical metric for the protocol:

$$u = \frac{\#\text{ Borrowed tokens}}{\#\text{ Deposited tokens}} \in [0,1].$$

- The **lending rate** adjusts to control this ratio:

$$r^{-} = f^{IRS}(u).$$

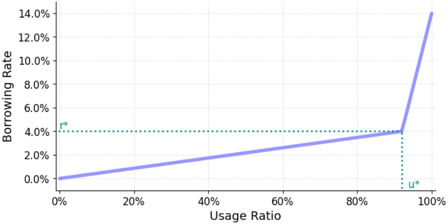


Figure: Interest Rate Strategy function for USDC. Snapshot date: 2026-04-12.

Interest Rates Strategy

- The share of interest τ allocated to the protocol is called the **reserve factor**.
- The remaining interest is distributed to lenders proportionally to their supplied amounts:

$$r^{+} = (1 - \tau) \cdot u \cdot r^{-}.$$

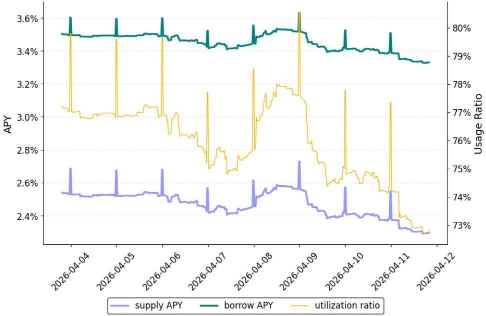


Figure: Historical lending and borrowing rates for USDC. From 2026-04-03 to 2026-04-12.

Historical Interest Rates

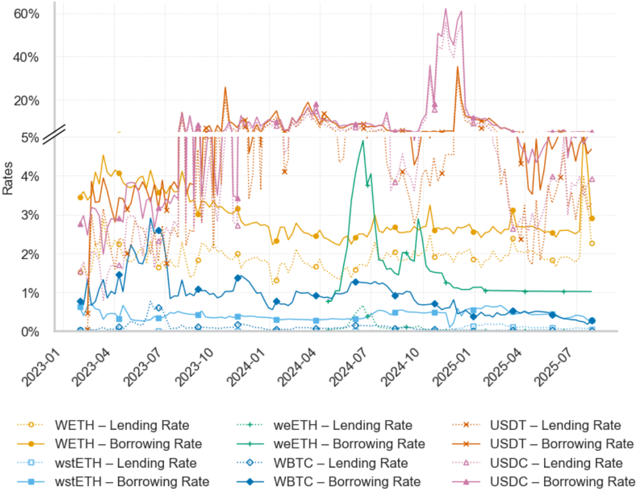


Figure: Historical Lending and Borrowing rates. From 2023-01-27 to 2025-08-01.

Over-Collateralized Loans

- The market prices of the collateral and borrowed tokens fluctuate over time.
- The protocol must ensure that borrowers' positions remain over-collateralized over time.
- A single metric captures the collateralization level: the **health factor (HF)**.

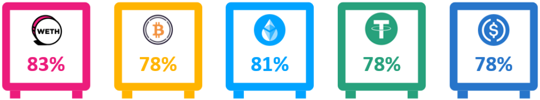
$$\begin{cases} \text{HF} > 1 & \text{Ok,} \\ \text{HF} \simeq 1 & \text{Danger,} \\ \text{HF} < 1 & \text{Liquidation.} \end{cases}$$

Over-Collateralized Loans

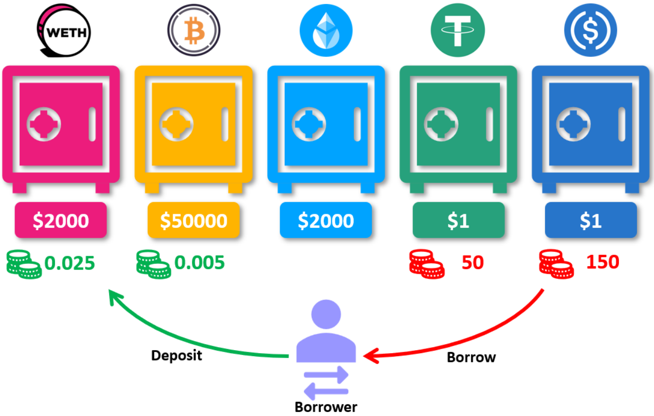
- HF is a collateral-to-debt ratio, with a penalty applied to the collateral value to enforce over-collateralization:

$$\mathrm{HF} = \frac{\sum_i \text{Collateral in reserve } i \times \mathrm{LT}_i}{\text{Total Debt}}$$

- LT is the **Liquidation Threshold**.
- Each asset has its own liquidation threshold.



Over-Collateralized Loans : Example

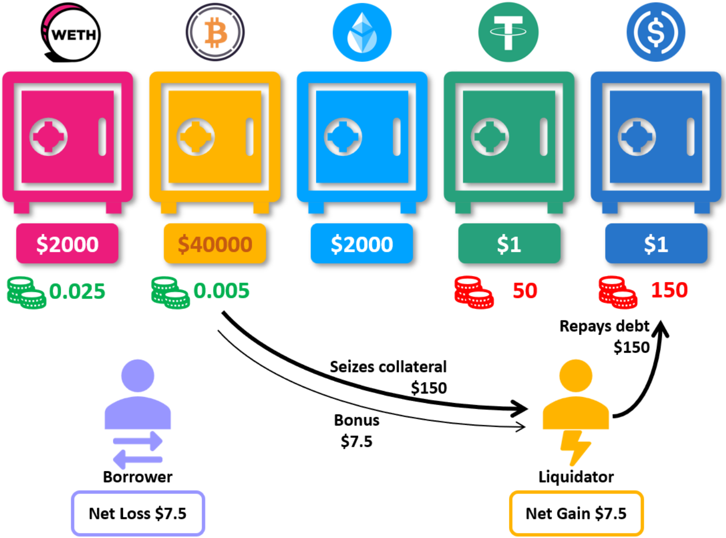


HF = ($50 · 83% + $250 · 78%) / ($50 + $150) = 1.18 ✓

Liquidations

- Liquidations can occur if HF ≤ 1.
- Permissionless (in practice, carried out by bots).
- During a liquidation, the liquidator:
  1. Repays the borrower's debt;
  2. Seizes the equivalent amount of collateral;
  3. Receives a bonus from the borrower's collateral.
- **Net gain for the liquidator, net loss for the borrower.**
- Liquidator's bonus determined by the **Liquidation Bonus** (LB) parameter.
- Each asset has its own liquidation bonus.

Liquidations: Example



Liquidations in Practice

24,374
Liquidations

$1,86B
Liquidated Debt

$1,96B
Seized Collateral

325
Distinct Liquidators

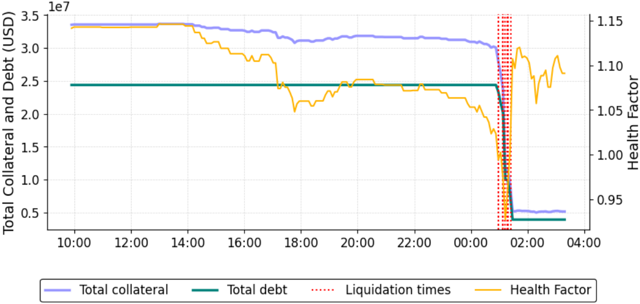
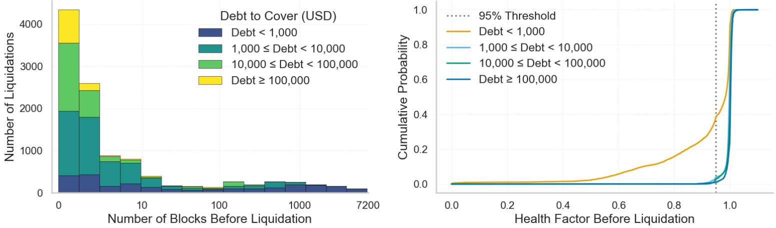


Figure: Real Liquidation Case Study. Date: 2024-08-05.

Liquidations in Practice



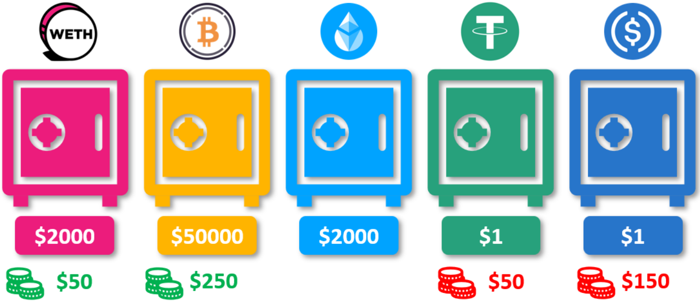
(a) Distribution of the number of blocks between the health factor dropping below one and the liquidation event.

(b) Cumulative distribution of health factors one block before liquidation.

Figure: Liquidation Efficiency

Outline

Notations



$2000	$50000	$2000	$1	$1
$50	$250		$50	$150
$\pi = 0.5$	$\pi = 2.5$	$\pi = 0$	$\pi = -0.5$	$\pi = -1.5$

$V_0 = \$100$

Notations

- n is the number of reserves in the protocol (today, $n = 65$).
- π_i is the initial USD value of the deposit/borrow in reserve i, normalized to 1:

  $$\begin{cases} \pi_i > 0 & \text{for deposit,} \\ \pi_i < 0 & \text{for borrow,} \end{cases} \qquad \text{and} \qquad \sum_{i=1}^{n} \pi_i = 1.$$

  The collateral and borrowed amounts in reserve i are π_i^+ and π_i^-.
- The lending and borrowing rates of reserve i at time s are denoted $r_{i,s}^+$ and $r_{i,s}^-$. Unified notation:

  $$r_{i,s}^{(\pi_i)} = \begin{cases} r_{i,s}^+ & \text{if} \quad \pi_i \geq 0, \\ r_{i,s}^- & \text{else.} \end{cases}$$

- The price of reserve i's token at time s is denoted $p_{i,s}$.

User Balance

- Given an initial position $\boldsymbol{\pi}$, the balance in reserve i at time t is given by:

$$\pi_{i,t} = \pi_i \frac{p_{i,t}}{p_{i,0}} e^{\int_0^t r_{i,s}^{(\pi_i)} \mathrm{d}s}.$$

- The health factor is given by:

$$\mathrm{HF}_t = \frac{\sum_{i=1}^n \pi_i^+ \frac{p_{i,t}}{p_{i,0}} e^{\int_0^t r_{i,s}^{(\pi_i)} \mathrm{d}s} \cdot \mathrm{LT}_i}{\sum_{i=1}^n \pi_i^- \frac{p_{i,t}}{p_{i,0}} e^{\int_0^t r_{i,s}^{(\pi_i)} \mathrm{d}s}}.$$

- The first possible liquidation time is:

$$\tau = \inf\{t > 0 : \mathrm{HF}_t \leq 1\}.$$

Protocol Solvency Condition

Proposition (No bad debt condition, continuous case)

Suppose that

- *Liquidations are instantaneous;*
- *Prices are continuous;*
- *The liquidation threshold and liquidation bonus parameters satisfy*

$$\max_{1\leq i\leq n} LT_i \cdot \max_{1\leq j\leq n} LB_j \leq 1.$$

Then, the no bad debt condition holds:

$$\sum_{i=1}^{n} \pi_i^{+} \frac{p_{i,\tau}}{p_{i,0}} e^{\int_0^{\tau} r_{i,s}^{(\pi_i)} ds} \cdot \frac{1}{LB_i} \geq \sum_{i=1}^{n} \pi_i^{-} \frac{p_{i,\tau}}{p_{i,0}} e^{\int_0^{\tau} r_{i,s}^{(\pi_i)} ds}.$$

Protocol Solvency Condition

The first two assumptions yields $\mathrm{HF}_\tau = 1$, thus

$$\sum_{i=1}^{n} \pi_i^{+} \frac{p_{i,\tau}}{p_{i,0}} e^{\int_0^\tau r_{i,s}^{(\pi_i)} \mathrm{d}s} \cdot \mathrm{LT}_i = \sum_{i=1}^{n} \pi_i^{-} \frac{p_{i,\tau}}{p_{i,0}} e^{\int_0^\tau r_{i,s}^{(\pi_i)} \mathrm{d}s}.$$

Combining with the last assumption yields:

$$\begin{aligned}
\sum_{i=1}^{n} \pi_i^{+} \frac{p_{i,\tau}}{p_{i,0}} e^{\int_0^\tau r_{i,s}^{(\pi_i)} \mathrm{d}s} \cdot \frac{1}{\mathrm{LB}_i} &\geq \sum_{i=1}^{n} \pi_i^{+} \frac{p_{i,\tau}}{p_{i,0}} e^{\int_0^\tau r_{i,s}^{(\pi_i)} \mathrm{d}s} \cdot \frac{1}{\max_j \mathrm{LB}_j} \\
&\geq \sum_{i=1}^{n} \pi_i^{+} \frac{p_{i,\tau}}{p_{i,0}} e^{\int_0^\tau r_{i,s}^{(\pi_i)} \mathrm{d}s} \cdot \mathrm{LT}_i \\
&= \sum_{i=1}^{n} \pi_i^{-} \frac{p_{i,\tau}}{p_{i,0}} e^{\int_0^\tau r_{i,s}^{(\pi_i)} \mathrm{d}s}.
\end{aligned}$$

□

Protocol Solvency Condition

Proposition (No bad debt condition)

Suppose that

- *Liquidations are instantaneous;*
- *Prices jumps are bounded:*

$$c^{\downarrow}_{jump} \leq \frac{p_{i,t}}{p_{i,t^-}} \leq c^{\uparrow}_{jump};$$

- *The liquidation threshold and liquidation bonus parameters satisfy*

$$\max_{1\leq i\leq n} LT_i \cdot \max_{1\leq j\leq n} LB_j \leq \frac{c^{\downarrow}_{jump}}{c^{\uparrow}_{jump}}.$$

Then, the no bad debt condition holds.

Probability of Liquidation

- What is the probability of being liquidated before $T > 0$?
- Assumptions:
  - Instantaneous liquidations;
  - Prices evolve as a multivariate geometric Brownian motion:

    $$p_{i,t} = p_{i,0} e^{(\mu_i - \sigma_i^2/2)t + \sigma_i W_{i,t}},$$

    where $\boldsymbol{W}$ is a Brownian motion with correlation $\boldsymbol{C}$.
  - Interest rates are fixed over $[0, T]$.
- The liquidation time is given by:

$$\tau = \inf\{t > 0 : \mathrm{HF}_t \leq 1\}.$$

- We approximate the liquidation time by a simpler stopping time $\tau^{\text{app.}}$:

$$\tau^{\text{app.}} = \inf\{t > 0 : \langle \boldsymbol{\nu}, \boldsymbol{W}_t \rangle \leq -\delta - \lambda t\}.$$

First Liquidation Time Approximation

- Denote:

$$\mu_i^{(\pi_i)} = \mu_i + r_i^{(\pi_i)},$$

- **Linearization** of the Geometric Brownian motion:

$$\frac{p_{i,t}}{p_{i,0}} e^{r_i^{(\pi_i)} t} - 1 \simeq \sigma_i W_{i,t} + \mu_i^{(\pi_i)} t.$$

- We obtain:

$$\tau^{\text{lin.}} = \inf\{t > 0 : \langle \boldsymbol{\nu}^{\text{lin.}}, \boldsymbol{W}_t \rangle \leq -\delta^{\text{lin.}} - \lambda^{\text{lin.}} t\},$$

Proposition

We have, for any $p \geq 1$, as $T \to 0$:

$$\mathbb{E}\left[\left|\mathbb{1}_{\tau \leq T} - \mathbb{1}_{\tau^{lin.} \leq T}\right|\right] = O(T^{\frac{p}{p+1}}).$$

Second Liquidation Time Approximation

- Replace the arithmetic mean by the geometric mean:

$$\sum_{i=1}^{n} \alpha_i e^{(\mu_i^{(\pi_i)} - \sigma_i^2/2)t + \sigma_i W_{i,t}} \simeq e^{\sum_{k=1}^{n} \alpha_k \left[(\mu_k^{(\pi_k)} - \sigma_k^2/2)t + \sigma_k W_{k,t}\right]}.$$

- We obtain:

$$\tau^{\text{geo.}} = \inf\{t > 0 : \langle \boldsymbol{\nu}^{\text{geo.}}, \boldsymbol{W}_t \rangle \leq -\delta^{\text{geo.}} - \lambda^{\text{geo.}} t\},$$

Proposition

We have, for any $p \geq 1$, as $T \to 0$:

$$\mathbb{E}\left[\left|\mathbb{1}_{\tau \leq T} - \mathbb{1}_{\tau^{geo.} \leq T}\right|\right] = O(T^{\frac{p}{p+1}}).$$

Simulation Time Approximations: Empirical Validation

- Is the approximate liquidation time accurate in detecting liquidation ?
- **Simulation:** For $T \in \{1/6, 1/12, 1/52\}$, $n = 7$, and fixed μ, σ, r^-, r^+ estimated from the data, generate 1M price paths and measure the accuracy of the liquidation detection with $\tau^{\text{app.}}$.

Time	HF_0	$\mathbb{P}(\tau \leq T)$	Method	Accuracy (%)
1/6	1.15	0.4986	Geo. unb.	99.440 (0.015)
			Geo.	99.331 (0.016)
			Lin.	97.851 (0.028)
1/12	1.11	0.5060	Geo. unb.	99.651 (0.012)
			Geo.	99.593 (0.012)
			Lin.	98.468 (0.024)
1/52	1.05	0.5128	Geo. unb.	99.849 (0.008)
			Geo.	99.841 (0.008)
			Lin.	99.271 (0.017)

Table: Performance of the liquidation time approximations

Probability of Liquidation

Proposition

The probability of liquidation before time T is given by:

$$\mathbb{P}\left(\tau^{app.} \leq T\right) = g_T\left(\frac{\delta}{\sqrt{\boldsymbol{\nu}^\top \boldsymbol{C}\boldsymbol{\nu}}}, \frac{\lambda}{\sqrt{\boldsymbol{\nu}^\top \boldsymbol{C}\boldsymbol{\nu}}}\right),$$

where:

$$g_T(u, v) = e^{-2uv}\mathcal{N}\left(v\sqrt{T} - \frac{u}{\sqrt{T}}\right) + \mathcal{N}\left(-v\sqrt{T} - \frac{u}{\sqrt{T}}\right).$$

Liquidation Time: Expectation & Mode

Corollary

Conditional on the event $\{\tau^{app.} < +\infty\}$, *the mode* $m_{\tau^{app.}}$ *of the distribution of the liquidation time* $\tau^{app.}$ *is given by:*

$$m_{\tau^{app.}} = \begin{cases} \frac{-3+\sqrt{9+4\delta^2\lambda^2/\gamma^4}}{2\lambda^2}\gamma^2 & \text{if } \lambda \neq 0, \\ \frac{\delta^2}{3\gamma^2} & \text{if } \lambda = 0, \end{cases}$$

where $\gamma = \sqrt{\boldsymbol{\nu}^\top \boldsymbol{C}\boldsymbol{\nu}}$ *is introduced for notational convenience. The unconditional expectation of* $\tau^{app.}$ *satisfies*

$$\mathbb{E}\left[\tau^{app.}\right] = \begin{cases} -\delta/\lambda & \text{if } \lambda < 0, \\ +\infty & \text{if } \lambda \geq 0. \end{cases}$$

Impact of Liquidations on the Return Moments

- What is the impact of the liquidation risk on the expectation and variance of the return ?
- In the absence of liquidation, the return can be expressed as:

$$R_T(\boldsymbol{\pi}) = \sum_{i=1}^{n} \pi_i \frac{p_{i,T}}{p_{i,0}} e^{\int_0^T r_{i,s}^{(\pi_i)} \mathrm{d}s} - 1.$$

- In case of liquidation, we assume:
  - **Proportional collateral seizure:** the collateral is seized from each reserve, proportionally to the collateral available in this reserve at time of liquidation;
  - **Full repayment of the debt.**

User's balance after Liquidation

- What is the remaining amount of collateral of a user right after liquidation ?
- Denote $x_i \geq 0$ the amount of seized collateral in reserve i.
- Proportional collateral seizure: there exists $c > 0$ such that:

$$x_i = c \cdot V_0 \pi_i^+ \frac{p_{i,\tau}}{p_{i,0}} e^{r_i^{(\pi_i)} \tau}.$$

- Full repayment of the debt:

$$\sum_{i=1}^n x_i = V_0 \sum_{k=1}^n \pi_k^- \frac{p_{k,\tau}}{p_{k,0}} e^{r_k^{(\pi_k)} \tau}.$$

- Therefore c is the debt-to-collateral ratio at liquidation:

$$c = \frac{\sum_{k=1}^n \pi_k^- \frac{p_{k,\tau}}{p_{k,0}} e^{r_k^{(\pi_k)} \tau}}{\sum_{k=1}^n \pi_k^+ \frac{p_{k,\tau}}{p_{k,0}} e^{r_k^{(\pi_k)} \tau}} =: \Psi_\tau$$

User's balance after Liquidation

- After liquidation, the remaining amount of collateral in reserve i is thus:

$$\pi_{i,\tau} = V_0(1 - \Psi_\tau \mathrm{LB}_i)\pi_i^+ \frac{p_{i,\tau}}{p_{i,0}} e^{r_i^{(\pi_i)}\tau}.$$

- Combining the two cases, the return can be expressed as:

$$\begin{aligned} R_T(\boldsymbol{\pi}) = {} & \mathbb{1}_{\tau > T} \cdot \left[\sum_{i=1}^{n} \pi_i \frac{p_{i,T}}{p_{i,0}} e^{r_i^{(\pi_i)} T} - 1 \right] \\ & + \mathbb{1}_{\tau \leq T} \cdot \left[\sum_{i=1}^{n} (1 - \Psi_\tau \mathrm{LB}_i)\pi_i^+ \frac{p_{i,\tau}}{p_{i,0}} e^{r_i^{(\pi_i)} T} - 1 \right]. \end{aligned}$$

Approximation of the debt-to-collateral ratio Ψ_τ

Proposition

Define the process:

$$\psi_t := \frac{\sum_{k=1}^{n} \pi_k^+ e^{r_k^+ t} \frac{p_{k,t}}{p_{k,0}} LT_k}{\sum_{k=1}^{n} \pi_k^+ e^{r_k^+ t} \frac{p_{k,t}}{p_{k,0}}}.$$

In particular, the initial value ψ_0 is given by:

$$\psi_0 = \frac{\langle \boldsymbol{\pi}^+, \boldsymbol{LT} \rangle}{\langle \boldsymbol{\pi}^+, \mathbf{1} \rangle}.$$

Then, for any $p \geq 1$, as $T \to 0$, we have:

$$\left| \mathbb{1}_{\tau \leq T} (\Psi_\tau - \psi_0) \right|_{L^p} = O(\sqrt{T}).$$

Approximate Return

- We approximate the exact return:

$$
\begin{aligned}
R_T(\boldsymbol{\pi}) = \mathbb{1}_{\tau > T} \cdot & \left[\sum_{i=1}^{n} \pi_i \frac{p_{i,T}}{p_{i,0}} e^{r_i^{(\pi_i)} T} - 1 \right] \\
& + \mathbb{1}_{\tau \leq T} \cdot \left[\sum_{i=1}^{n} (1 - \Psi_\tau \mathrm{LB}_i) \pi_i^+ \frac{p_{i,\tau}}{p_{i,0}} e^{r_i^{(\pi_i)} T} - 1 \right],
\end{aligned}
$$

with the approximate return:

$$
R_T^{\text{app.}}(\boldsymbol{\pi}) = \sum_{i=1}^{n} \pi_i e^{r_i^{(\pi_i)} T} \frac{p_{i,T}}{p_{i,0}} - 1 + \mathbb{1}_{\tau^{\text{app.}} \leq T} \cdot \sum_{i=1}^{n} w_i e^{r_i^{(\pi_i)} T} \frac{p_{i,T}}{p_{i,0}},
$$

where

$$
w_i = (1 - \psi_0 \mathrm{LB}_i) \pi_i^+ - \pi_i.
$$

Technical Tool

Recall that:

$$g_T(u,v) = e^{-2uv}\mathcal{N}\left(v\sqrt{T} - \frac{u}{\sqrt{T}}\right) + \mathcal{N}\left(-v\sqrt{T} - \frac{u}{\sqrt{T}}\right),$$

Theorem

Let $\boldsymbol{B}$ be a n-dimensional standard Brownian motion. Given a correlation matrix $\boldsymbol{C}$, define the first hitting time

$$\rho := \inf\left\{t > 0 : \langle \boldsymbol{\nu}, \boldsymbol{C}^{1/2}\boldsymbol{B}_t\rangle \leq -\delta - \lambda t\right\},$$

where $\boldsymbol{\nu} \in \mathbb{R}^n$, $\delta > 0$ and $\lambda \in \mathbb{R}$. When $\boldsymbol{\nu}^\top \boldsymbol{C}\boldsymbol{\nu} > 0$, we have:

$$\mathbb{E}\left[e^{\langle \boldsymbol{a}, \boldsymbol{C}^{1/2}\boldsymbol{B}_T\rangle - \frac{T}{2}\boldsymbol{a}^\top \boldsymbol{C}\boldsymbol{a}}\mathbb{1}_{\rho \leq T}\right] = g_T\left(\frac{\delta}{\sqrt{\boldsymbol{\nu}^\top \boldsymbol{C}\boldsymbol{\nu}}}, \frac{\lambda + \boldsymbol{\nu}^\top \boldsymbol{C}\boldsymbol{a}}{\sqrt{\boldsymbol{\nu}^\top \boldsymbol{C}\boldsymbol{\nu}}}\right),$$

for any $\boldsymbol{a} \in \mathbb{R}^n$.

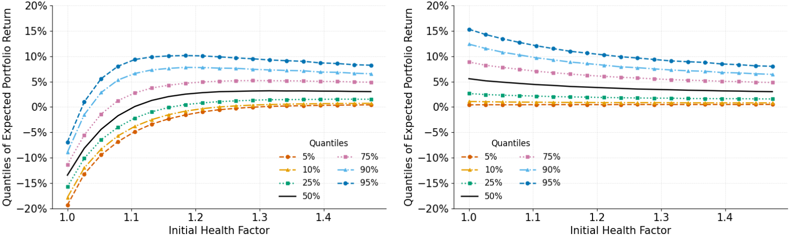
Approximate Return

- As a consequence, all the moments of the approximate return:

$$R_T^{\text{app.}}(\boldsymbol{\pi}) = \sum_{i=1}^{n} \pi_i e^{r_i^{(\pi_i)} T} \frac{p_{i,T}}{p_{i,0}} - 1 + \mathbb{1}_{\tau^{\text{app.}} \leq T} \cdot \sum_{i=1}^{n} w_i e^{r_i^{(\pi_i)} T} \frac{p_{i,T}}{p_{i,0}},$$

  can be expressed with a combination of $g_T(\cdot, \cdot)$.
- In particular, we obtain a closed form for the expectation and variance.

Application: Impact of Liquidations on Expected Returns

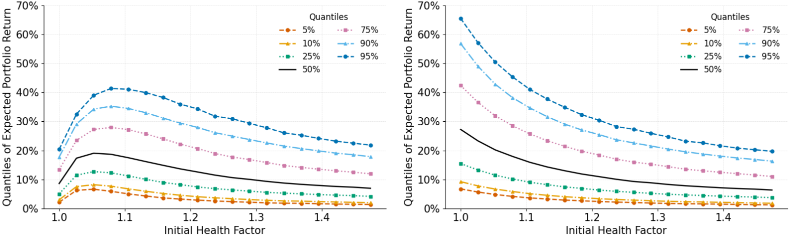


(a) Quantiles of expected portfolio return under the liquidation-aware framework

(b) Quantiles of expected portfolio return under the liquidation-ignoring framework

Figure: Empirical quantiles of portfolio return expectations, conditional on the initial health factor. The sample consists of 1M portfolios randomly generated such that their initial health factor exceeds one and their expected return under the liquidation-ignoring framework is positive.

Application: Impact of Liquidations on Returns Variance



(a) Quantiles of portfolio return variance under the liquidation-aware framework

(b) Quantiles of portfolio return variance under the liquidation-ignoring framework

Figure: Empirical quantiles of portfolio return variances, conditional on the initial health factor. The sample consists of 1M portfolios randomly generated such that their initial health factor exceeds one and their expected return under the liquidation-ignoring framework is positive.

Application to fast sampling scheme

Input: Portfolio vector $\boldsymbol{\pi}$, liquidation bonuses vector **LB**, volatility vector $\boldsymbol{\sigma}$, correlation matrix $\boldsymbol{C}$, liquidation parameters $\boldsymbol{\nu}$, λ and δ.

Output: A random sample of $(\tau^{\text{app.}}, R_T^{\text{app.}}(\boldsymbol{\pi}))$.

1. Simulate the liquidation time $\tau^{\text{app.}}$ via the inverse CDF method.

Application to fast sampling scheme

Case A: $\tau^{\text{app.}} \leq T$ **(Liquidation occurs)**:

2. Set $\widehat{B}_{1,\tau^{\text{app.}}} = -\delta/\gamma - \lambda\tau^{\text{app.}}/\gamma$;
3. Sample the remaining Brownian components at $\tau^{\text{app.}}$:

$$\left(\widehat{B}_{2,\tau^{\text{app.}}}, \ldots, \widehat{B}_{n,\tau^{\text{app.}}}\right) \sim \mathcal{N}\left(0, \tau^{\text{app.}} \boldsymbol{I}_{n-1}\right);$$

4. Extend the path to the horizon T, using the strong Markov property of the Brownian motion:

$$\widehat{\boldsymbol{B}}_T \sim \mathcal{N}\left(\widehat{\boldsymbol{B}}_{\tau^{\text{app.}}}, \, (T - \tau^{\text{app.}})\boldsymbol{I}_n\right);$$

5. Recover $\boldsymbol{W}_T$ and compute the return:

$$R_T^{\text{app.}}(\boldsymbol{\pi}) = \sum_{i=1}^{n} (1 - \psi_0 \text{LB}_i)\, \pi_i^+ \, e^{\left(\mu_i^{(\pi_i)} - \sigma_i^2/2\right) T + \sigma_i W_i} - 1.$$

Application to fast sampling scheme

Case B: $\tau^{\textbf{app.}} > T$ (No liquidation)

2. Simulate the first Brownian component $\widehat{B}_{1,T}$ conditional on the no-liquidation event

$$\left\{ \inf_{t \leq T} \gamma \widehat{B}_{1,t} + \lambda t + \delta > 0 \right\},$$

which can be sampled via the inverse CDF method;

3. Sample the other Brownian components at time T:

$$\left(\widehat{B}_{2,T}, \dots, \widehat{B}_{n,T} \right) \sim \mathcal{N}\left(0,\ T\boldsymbol{I}_{n-1}\right);$$

4. Recover $\boldsymbol{W}_T$, then the approximate portfolio return is

$$R_T^{\text{app.}}(\boldsymbol{\pi}) = \sum_{i=1}^{n} \pi_i \, e^{\left(\mu_i^{(\pi_i)} - \sigma_i^2/2\right) T + \sigma_i W_i} - 1.$$

Portfolio Optimisation

- Objective function to minimize:

$$\ell(\boldsymbol{\pi}) = \frac{\phi}{2}\left(\mathbb{E}\left[R(\boldsymbol{\pi})^2\right] - \mathbb{E}\left[R(\boldsymbol{\pi})\right]^2\right) - \mathbb{E}\left[R(\boldsymbol{\pi})\right]$$

- Continuous, orthant-wise differentiable, but non-convex and non-differentiable at each orthant border.
- First-order optimisation algorithms, where the gradients are computed via finite differences.

Portfolio Optimisation

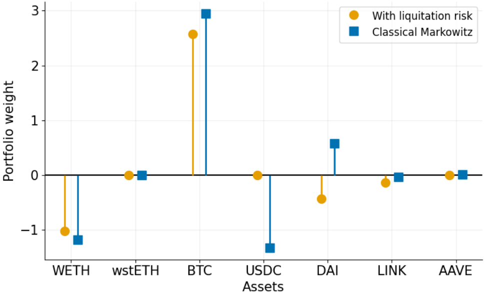


Figure: Comparison of optimal portfolios under the liquidation-aware framework and the classical Markowitz framework.